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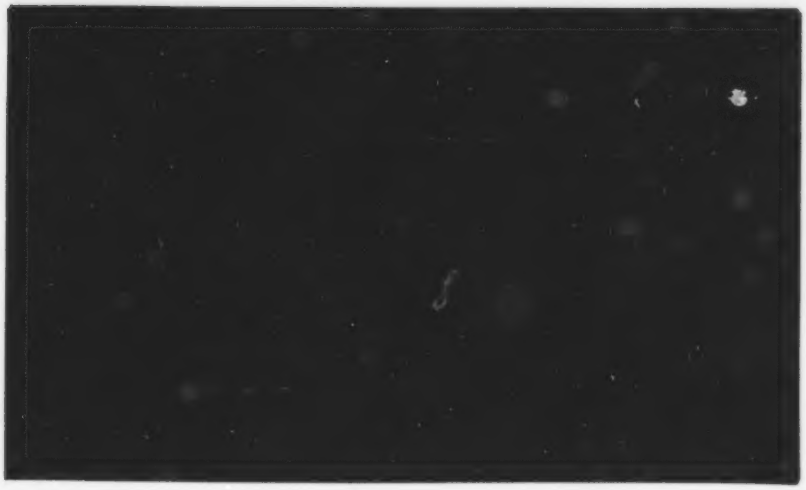

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AN OPTIMAL ALGORITHM FOR WEIGHTED MINIMAX FLOW CENTERS ON TREES

by

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ABSTRACT

Let a tree be given with supplies assigned to its nodes and weights assigned to its edges. The problem considered here is to locate a facility, so that the flow pattern which results from shipping all supplies to the facility, is such that it minimizes the largest weighted flow on any individual edge. First some properties of such facility locations are derived and then a linear-time algorithm for their determination is given.

Key words : facility location; minimax flow problem; location on trees.

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RÉSUMÉ

Étant donné un arbre, des offres d'un produit aux sommets de cet arbre et des poids associés aux arêtes, nous considérons le problème de la détermination de centres de flot pondéré minimax sur cet arbre, c'est-à-dire de sommets tels que si toutes les quantités du produit présentes dans l'arbre sont envoyées à l'un ou l'autre de ces sommets, le plus grand flot pondéré sur une arête soit minimisé. Nous démontrons d'abord certaines propriétés relatives aux centres de flot pondéré minimax, puis nous décrivons un algorithme en temps linéaire pour leur détermination.

Mots-clés : localisation de ressource; problèmes de flot minimax; localisation de ressource.

INTRODUCTION

In this paper, the following problem is considered. Let a tree be given with supplies at its vertices and nonnegative weights assigned to its edges. The objective is to determine a vertex v* with the following property. If the supplies of all vertices are shipped on their unique paths to v*, then the largest weighted edge flow is as small as possible. A vertex v* with this property is called a **weighted minimax flow center**, or **WMMFC**. The major contribution of this paper is the description of a linear-time algorithm for the problem. The method described here generalizes earlier results by Eiselt and Gendreau (1989) for minimax flow centers with all weights equal.

The problem as outlined above has a large number of applications. Here, we briefly describe a stochastic two-player game which can readily be solved by using this model.

Let the given tree symbolize the communication network of one of the players (the "Defender"). The vertices denote points at which messages originate and the edges existing communication links; traffic intensity between vertices and headquarters are indicated by supplies, while the weight on each edge corresponds to the probability that the edge will be destroyed if it is attacked by the other player (the "Attacker". The objective of the Defender is to locate headquarters at a vertex in such a way as to minimize the expected loss of traffic which will be incurred if the network is attacked. On the other hand, the goal of the Attacker is to maximize this expected loss. If one assumes that the Attacker is able to mount a single attack within a specified time horizon and that he has a perfect knowledge of network configuration, traffic intensities, and headquarters location, this attack should occur on the link for which the weighted flow is maximum. To optimally locate his headquarters, the Defender should thus solve the corresponding WMMFC problem.

An interesting special case of this problem occurs when the network is made up of two types of edges, viz. vulnerable and invulnerable edges. Minimizing the largest flow in a vulnerable edge is done by using unit weights for vulnerable and zero weights for invulnerable edges. In case of multiple optimal locations, secondary criteria could be used.

Obviously, one could also use the WMMFC framework to model similar situations in which the network under attack is not a communication but rather a transportation or logistic network.

Minimax problems are well-known in the literature. As far as location models are concerned, they are to be distinguished from center (i.e. bottleneck) problems where the largest attribute (i.e. weight) of an edge is sought among all edges which carry any flow; subsequently, this largest attribute is to be minimized. In contrast to that, in our problem the flow is part of the function to be minimized. Minimax flow centers were introduced by Eiselt and Pederzoli (1984). To our knowledge, the first algorithm for minimax flow problems was provided by Minoux (1976) and a polynomial method was put forward by Ahuja (1986). Some aspects of minimax flows in bipartite graphs were investigated by Eiselt (1988) and general minimax linear programming problems were solved by Ahuja (1985).

THE MODEL AND ITS PROPORTIONS

Let $T=(V, E)$ be a tree with set of vertices $V=\{v_1, v_2, \ldots, v_n\}$ and set of edges $E=\{e_1, e_2, \ldots, e_m\}$. The supplies (or similarly, demands) are denoted by $s(v_i) > 0, \quad \forall v_i \varepsilon V$. The supply of a subtree $\bar{T}$ of T is $s(\bar{T}) = \sum_{v_i \text{ in } \bar{T}} s(v_i)$ and for simplicity, $S=s(T)$. The weights associated with the edges are $c_k > 0, \quad \forall e_k \varepsilon E$. Furthermore, let $f^*=\{f^*(e_k)\}$ denote the unique flow in T given that the facility to which all supplies are shipped is located at v^*. We can then define an (unweighted) minimax flow center (MMFC) as a vertex v^* which minimizes

$$z_u^* = \max_{e_\ell \in E} \{ f^*(e_\ell) \}$$

whereas the weighted minimax flow center WMMFC is a vertex v* minimizing

$$z_w^* = \max_{e_\ell \in E} \{ c_\ell \; f^*(e_\ell) \}$$

Futhermore, we will call e_k **a critical edge,** if

$$e_k \in \operatorname*{argmax}_{e_\ell \in E} \{ c_\ell \; f^*(e_\ell) \}$$

and similarly for the unweighted case. It should be pointed out that critical edges in the unweighted problem are always adjacent to the MMFC whereas this is not necessarily the case in the weighted model. In other words, unweighted flows accumulate towards the facility location v* whereas weighted flows do not. This can be visualized in Figures 1a and 1b with Figure 1a showing the tree with all supplies and edge weights whereas Figure 1b displays the weighted edge flows towards the unique MMFC and WMMFC at v_4 (with the edge connecting v_9 and v_5 being critical).

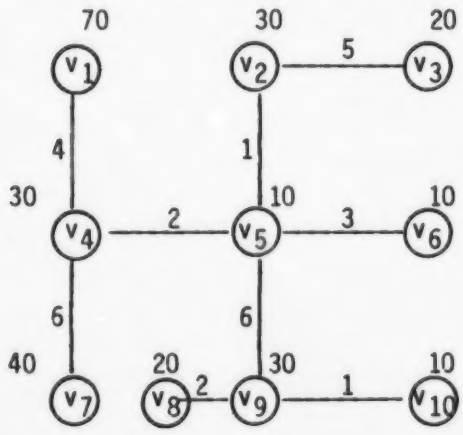

Figure 1a

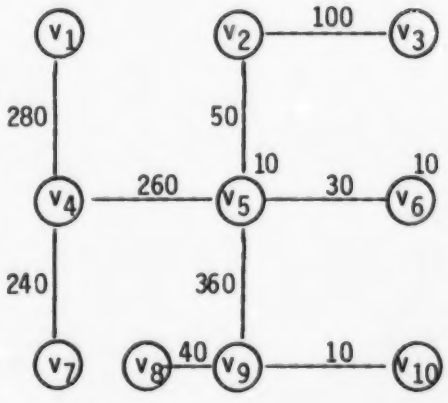

Figure 1b

We are now able to state two lemmas concerning the location(s) of WMMFC(s) in a tree.

Lemma 1: Each MMFC of T is also a WMMFC for any edge weight $c_\ell > 0$, $\forall e_\ell \in E$.

Proof: Suppose that v* is an MMFC but not a WMMFC. Let v' be a WMMFC, so that

$$z'_w = \max_{e_\ell \in E} \{c_\ell \; f'(e_\ell)\} < \max_{e_\ell \in E} \{c_\ell \; f^*(e_\ell)\} = z^*_w \; . \qquad (1)$$

Denote the unique path between v* and v' by P(v*, v'). Note that $f'(c_\ell) = f^*(e_\ell)$, $\forall e_\ell \notin P(v^*, v')$. Furthermore, $f'(e_\ell) = S - f^*(e_\ell)$, $\forall e_\ell \in P(v^*, v')$. The assumption that v* is an MMFC implies that $f^*(e_\ell) < S/2$, $\forall e_\ell \in E$ (see Eiselt and Gendreau, 1989). Hence $f'(e_\ell) > S/2$, $\forall e_\ell \in P(v^*, v')$ and thus

$$f'(e_\ell) > f^*(e_\ell), \; \forall e_\ell \in E. \qquad (2)$$

As $c_\ell > 0$, $\forall e_\ell \in E$, condition (2) implies that

$$c_\ell \; f'(e_\ell) > c_\ell \; f^*(e_\ell), \; \forall e_\ell \in E$$

and therefore

$$z'_w = \max_{e_\ell \in E} \{c_\ell \; f'(e_\ell)\} > \max_{e_\ell \in E} \{c_\ell \; f^*(e_\ell)\} = z^*_w$$

which contradicts condition 1. Consequently, v* must be a WMMFC. ∎

On the other hand, it can be shown that the converse is not true. Consider the tree in Figure 2 where the numbers next to the vertices represent their respective weights and those next to the edges indicate their weights.

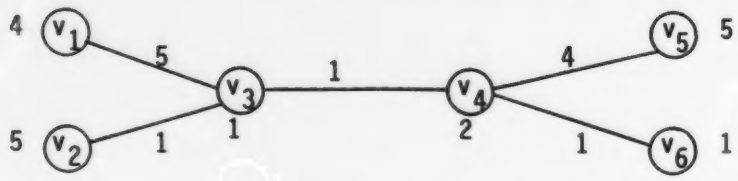

Figure 2

whereas v_3 is the unique MMFC with $z_u^* = 8$; v_2, v_3, v_4 and v_6 are all WMMFCs with edges (v_1, v_3) and (v_5, v_4) carrying a weighted flow of 20 and being thus critical in all cases.

Consequently, if we were only interested in any of the WMMFCs, any algorithm which finds MMFCs could be employed to solve the problem. In the following, we describe properties of the set of all WMMFCs which are then used in the next section in an algorithm. In the ensuing discussion, let v^* be the location of an MMFC with associated flow pattern f^* and weighted objective value z_w^*. Furthermore, consider an arbitrary edge $e_k \varepsilon E$ whose removal from E subdivides T into two subtrees T_k^1 and T_k^2. Without loss of generality, assume that v^* is in T_k^1. If a WMMFC were to be located at any vertex v^* in T_k^2, then a supply of $S(T_k^1)$ would flow along edge e_k, so that $z_w'' > c_k\ f''(e_k) = c_k\ S(T_k^1)$. In case that $c_k\ s(T_k^1) > z_w$, $z_w'' > z_w^*$ follows and v'' cannot be a WMMFC. If this occurs, the edge e_k will be called <u>blocking</u>. Note that if the flow pattern f'' is known, the flow $f(e_k)$ can easily be computed. As $f^*(e_k) = s(T_k^2)$, $f''(e_k) = S(T_k^1) = S\text{-}s(T_k^2) = S\text{-}f^*(e_k)$.

Consider now some vertex $v'' \neq v^*$ and recall that the flow patterns f'' and f^* differ only on edges on $P(v^*, v'')$; see the proof of lemma 1. Define now a vertex $v' \varepsilon V \setminus \{v''\}$ as a predecessor of v'' if $v' \varepsilon P(v^*, v'')$ and there is an edge $e'=(v', v'')$. Note that such a vertex always exists (when v'' and v^* are adjacent, $v'=v^*$). Also observe that v' is unique. We can then state

Lemma 2: A vertex $v'' \neq v^*$ is a WMMFC if and only if its predecessor v' is a WMMFC and the edge $e'=(v', v'')$ is not blocking.

Defining V^* as the set of all WMMFCs and E^* the set of non-blocking edges incident to vertices in V^*, lemma 2 implies

Corollary 3: The graph $T^*=(V^*, E^*)$ is a connected subtree of T which contains v^*.

In the next section, we show how the set V^* can be determined efficiently.

AN ALGORITHM FOR THE DETERMINATION OF ALL WMMFCs

An immediate algorithmic consequence of lemma 2 is that it makes it possible to find the set of WMMFCs by examining each edge with respect to its blocking property. This will be done in a systematic way. The basic idea of the algorithm is as follows. First, an MMFC along with its flow and objective value is determined with an appropriate algorithm. Starting with an unweighted minimax flow center as the only vertex in V^*, the algorithm constructs the set V^* by successively including vertices which are neighbors of elements of V^*, if the connecting edge is non-blocking. A formal statement of the procedure is provided below. Here, V^* denotes the set of currently known WMMFCs and $U \subset V^*$ consists of those vertices whose neighbors have not yet been checked as far as their inclusion in V^* is concerned.

Algorithm: Weighted Minimax Flow Centers in Trees

Step 1: Find an MMFC of T, say v^*, and compute the associated flow pattern f^* and objective value z^*. Compute S.

Step 2: If these have been modified by step 1, re-initialize the edge sets $E(v_i)$ of all vertices v_i, $v_i \in V$, and set

$$V^* \leftarrow \{v^*\},$$
$$U \leftarrow \{v^*\}.$$

Step 3: (Processing of unchecked WMMFCs)

While $U \neq \emptyset$ do

[pick some $v' \in U$,
[set $U \leftarrow U \setminus \{v'\}$,
[perform step 4.

Stop, the set V^* now includes all WMMFCs.

Step 4: (Checking neighbors of v')

While $E(v') \neq \emptyset$ do

[pick some $e_k = (v', v_k)$ in $E(v')$,
[set $E(v') \leftarrow E(v') \setminus \{e'\}$,
[set $E(v_k) \leftarrow E(v_k) \setminus \{e'\}$,
[if $c_k(S - f^*(e_k)) < z^*$, then e_k is non-blocking,
[set $V^* \leftarrow V^* \cup \{v_k\}$,
[set $U \leftarrow U \cup \{v_k\}$.

Two notes are in order with regards to the implementation of this algorithm. First, to allow for the deletion of edges from edge lists $E(v_i)$ in constant time, these should be organized as doubly-linked lists. Secondly, it would be convenient to use a stack for set U.

The complexity of the algorithm can now be evaluated as follows:

- Step 1 can be performed in linear-time (i.e. O(n)) by slightly modifying the Eiselt-Gendreau (1989) procedure for MMFC to account for the need to fully compute f^*.

- Step 2 requires at most O(n) operations, if properly implemented.
- Step 3 is executed at most once for each vertex; not accounting for the operations performed in step 4, each execution requires a constant number of operations.
- The inner loop of step 4 is performed at most (n-1) times: once per edge of T. This loop can also be executed in constant time, yielding an overall complexity of O(n) operations for this step.

The algorithm has thus a worst-case complexity of O(n). Since each of the O(n) parameters of the problem has to be considered at least once by any algorithm, this is also a lower bound on the complexity. The algorithm proposed here is therefore optimal except maybe for a constant factor.

CONCLUSION

In this paper, the problem of finding weighted minimax flow centers in trees has been discussed. First, it was proved that each unweighted minimax flow center is also a WMMFC but not vice versa. Then a linear-time algorithm for the determination of WMMFCs has been described.

Many extensions of the problem studied here can be thought of. One possibility is to consider minimax flow centers in general graphs even though it appears that this problem requires the use of a minimax flow algorithm which was unnecessary for the problem studied here. This would dramatically increase the computational complexity of the problem. Other possible generalizations include multicommodity minimax flow centers as well as (W)MMFCs with gains.

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